Document extract

Title of chapter/article	Developing an Understanding of the Size of Fractions
Author(s)	Peter Gould
Copyright owner	The Australian Association of Mathematics Teachers (AAMT) Inc.
Published in	Fractions: Teaching for Understanding
Year of publication	2011
Page range	63–70
ISBN/ISSN	978-1-875900-68-8

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CHAPTER 6.

DEVELOPING AN UNDERSTANDING OF THE SIZE OF FRACTIONS*

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False friends (*faux amis)* are two words that look and sound similar yet have unlike meanings in different languages. Over 30 years ago, Richard Skemp (1978) used the analogy of false friends to explain how the same word can be interpreted with diverse meanings even in the same language. In particular, he described two different meanings people appeared to apply when describing mathematical understanding. One meaning given to mathematical understanding is having a rule and being able to use it. The second meaning, which Skemp described as relational understanding, is knowing both what to do and why it works. Within the classroom, *faux amis* exist when we address understanding the size of fractions.

Finding out what students think

Asking the right question is pivotal to finding out what students think. If we only ask students to colour in specific fractions of pre-divided shapes, as is common in many text series, we may not discover what students think about fractions. Questions such as "Colour in the correct number of equal parts to show $\frac{3}{8}$ of the following shape" do not require any appreciation of the size of fractions to complete.

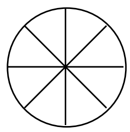

Figure 6.1

* The chapter is based on research findings presented in Gould, P. (2005). Year 6 students' methods of comparing the size of fractions. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Prierce & A. Roche (Eds). *Building connections: Research, theory and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia Inc. (pp. 393–400). Melbourne: MERGA.

Rather than demonstrating an understanding of the size of fractions, the above task requires only simple counting to colour someone else's representation of a fraction.

Asking students to explain their reasoning when comparing the size of fractions produces greater insights into their current understanding of fractions. In a study of students' understanding of the size of fractions, 100 Year 6 students were asked to determine the larger of two fractions ($\frac{1}{2}$ or $\frac{1}{3}$, $\frac{1}{4}$ or $\frac{1}{5}$, $\frac{1}{5}$ or $\frac{1}{6}$, $\frac{1}{6}$ or $\frac{1}{12}$, $\frac{1}{6}$ or $\frac{1}{3}$, $\frac{2}{3}$ or $\frac{5}{6}$, $\frac{9}{10}$ or $\frac{12}{13}$) and to explain their reason for the decision. The questions were read out and most students used diagrams as either part or the whole of their explanation. However, it was what they did with the diagram that was most telling about their understanding.

Sometimes when a student attempted to make use of an area model in his or her explanation, it became clear, as in Figure 6.2, that the representation did not reflect equal areas.

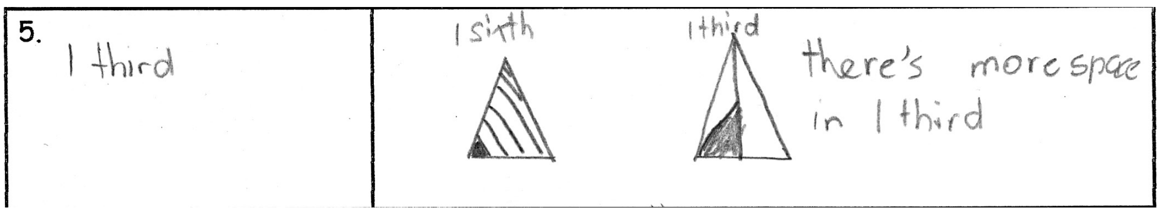

Figure 6.2. Which is bigger, one-third or one-sixth?

In Figure 6.2, the area of the pieces used to represent the fractions is not the apparent focus of the student's attention. The number of pieces appears to define the fraction for this student, despite the appeal to the amount of "space" to justify the answer. Having students represent fractions *as they think about them* is important in determining what students have learnt.

In a number of responses it became clear that students were dealing with the number of parts when they drew subdivided shapes, rather than the relative area of the parts. In determining which is larger, $\frac{1}{6}$ or $\frac{1}{3}$, a student (Figure 6.3) indicated that $\frac{1}{6}$ was larger after subdividing two circles and marking three and six parts. This representation was clearly about the number of parts rather than the area of the parts. Moreover, the number of parts indicated by the student, corresponded to the numerical value of the denominator.

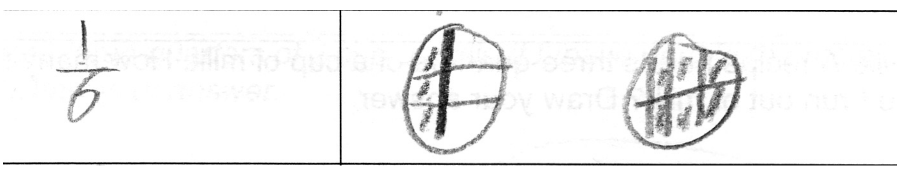

Figure 6.3. Representing a number of parts rather than the area of the parts.

Equal wholes

Some diagrams also provided an indication that students were not always aware of the need to have equal wholes to compare fraction quantities.

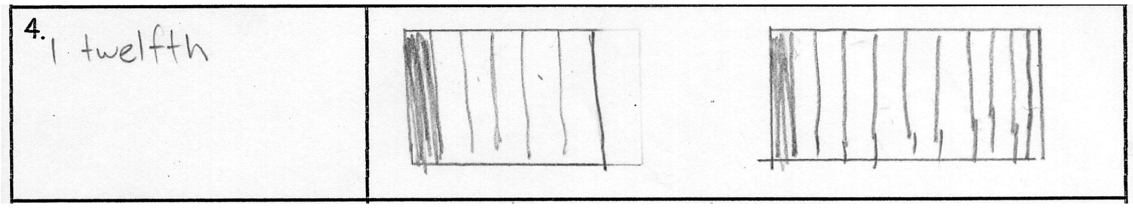

Figure 6.4. Which is bigger, one-sixth or one-twelfth?

The student's diagram shown in Figure 6.4, offered as an explanation for why one-twelfth is bigger than one-sixth, shows one part out of six shaded and for one-twelfth, a slightly smaller part is shaded out of a significantly larger unit-whole rectangle. It is conceivable that it is the increased size of the whole that becomes the basis for deciding which is the bigger fraction. This interpretation is aligned to the findings of a study by Yoshida and Kuriyama (1995) where many students drew representations in which the size of the whole each fraction represented, was in direct proportion to the size of the denominator. This "growing whole" was most likely to appear where students used shaded rectangles to represent the fractions. In comparing two-thirds and five-sixths it is possible to arrive at a correct conclusion (that five-sixths is larger) by incorrectly increasing the size of the unit whole (see Figure 6.5).

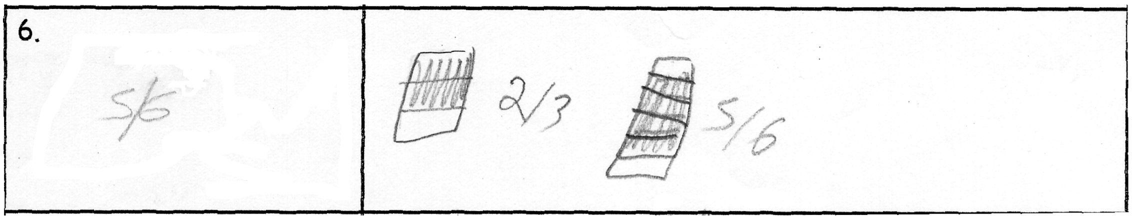

Figure 6.5. Five-sixths showing an increased unit-whole.

The limits of rule-based decisions

For the first five questions, students could consistently apply the rule, "the bigger the denominator the smaller the fraction" or the "greater the number of pieces the smaller the pieces". As one student eloquently recorded in response to the third question, "…as the denominator gets bigger it gets further away from 1". Question six and question seven necessitated a change of comparison strategy, as the fractions being compared were no longer unitary fractions. Although the percentages of correct answers suggested that the majority of the students could compare fractions, many of their correct answers were based on faulty reasoning.

Questions six and seven were designed to prompt a shift in comparison strategies. Although a focus on the numeric size of the denominator, including a "bigger means smaller interpretation", was possible with the first five questions, this strategy would be inadequate for the final two questions. These questions compared fractions that were one fractional part smaller than one whole. Instead of comparing one-third and one-sixth as in question five, question six compared their complements, i.e. two-thirds and five-sixths (see Figure 6.5). With question six, students could reasonably argue in terms of common denominators, converting two-thirds to four-sixths. Alternatively, they could argue proximity to 1 using the information on the relative size of one-third and one-sixth. Reasoning based on how close each fraction is to 1 whole draws on a quite sophisticated understanding of the size of fractions. Three students argued for their answer to question 6 based on the gap to 1 whole.

The comparison of $\frac{9}{10}$ and $\frac{12}{13}$ provided some fascinating insights. Two students adopted a purely additive strategy and commented that you could go from $\frac{9}{10}$ to $\frac{12}{13}$ by adding 3 to the top and the bottom. One concluded that $\frac{12}{13}$ was bigger because it was 3 more (on the top and bottom) while another argued that this made the fractions the same size. In total, seven per cent of students argued that the final two fractions were equal. The belief that the two fractions ($\frac{9}{10}$ and $\frac{12}{13}$) are the same size does not diminish in high school. In a much larger study involving equal numbers of Year 7 and 8 students (n = 684), 8% also argued that the two fractions were the same size.

Disconnected ideas

The explanation that the higher the number on the bottom the smaller the fraction, was very common. For example, one student consistently applied the explanation "if the bottom number is bigger, it is smaller" to explain the responses to the first five questions. In addition to the "bigger is smaller" explanation, the student drew and labelled partitions of squares, rectangles and circles. The diagrams frequently showed area models that did not allow comparisons, as they did not depict equal wholes or equal partitioning (see Figures 6.6 and 6.7).

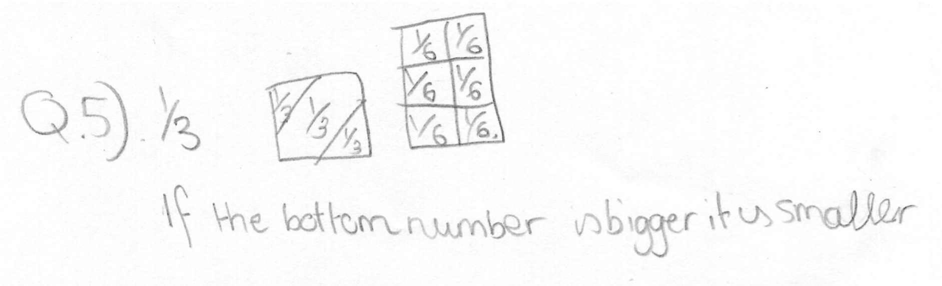

Figure 6.6. Neither equal-area partitioning nor equal wholes.

The student whose responses are depicted in Figures 6.6 and 6.7 does not acknowledge the need for equal wholes nor use consistent shapes to represent the whole. In Figure 6.7, the student partitions a circle into five equal parts and a rectangle is partitioned into sixths to "compare" the fractions one-fifth and one-sixth.

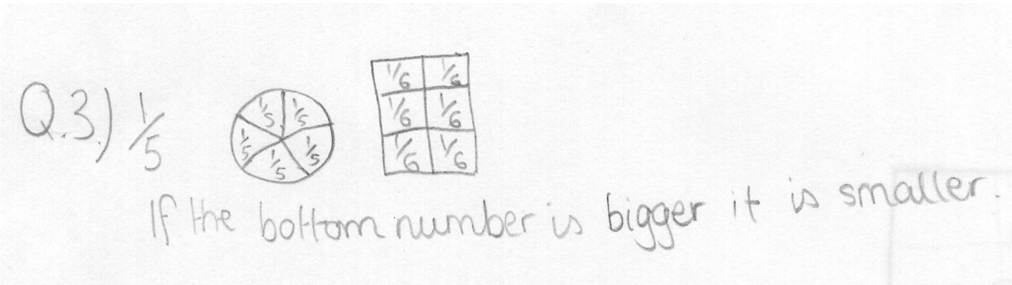

Figure 6.7. A pictorial comparison of fifths and sixths in different shapes.

What is clear from these responses is that although drawings are used in the explanation, area cannot be the feature being used by the student to answer the questions. Although the student has learnt to draw and label subdivided shapes, the activity of drawing and labelling does not link to the intended meaning of the drawings. Rules are being learnt for handling fractions but these rules are often not connected to the meaning of fractions as mathematical quantities.

In direct contrast to this explanation was the reasoning that as the denominator got bigger the fraction was considered to be bigger. A focus on the value of the whole number written as the denominator meant that some students interpreted the size of the denominator as being proportional to the size of the fraction (see Figure 6.8).

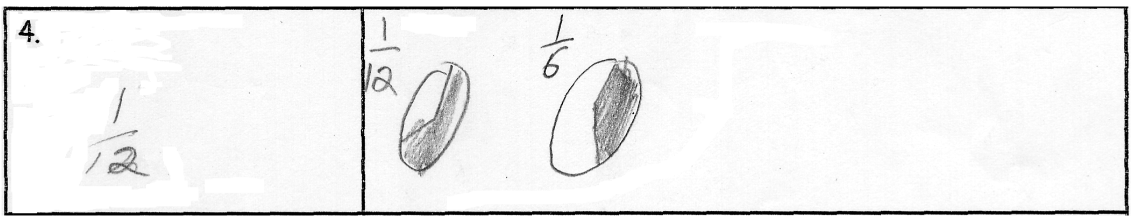

Figure 6.8. Fractions portrayed as "the larger the denominator the larger the fractional part".

In this instance the drawings link to an incorrect interpretation of the size of fractions. However, interpreting the size of the fraction as being proportional to the size of the whole number represented by the denominator did not always link to area diagrams. Sometimes explanations referred directly to the size of the denominators: "because 12 is higher than six".

What students learn is not always what we thought we taught

Most teaching programs for fractions use area models and emphasise the need for equal parts. However, it is not always clear exactly what attribute is being considered when teachers and students refer to equal parts. Students' explanations relating to the size of fractions when accompanied by drawings sometimes appear to be about the number of parts rather than comparisons of area.

To be able to interpret the part-whole comparison of area intended by the regional model, students need to be familiar with the context, which for regional models includes the concept of area. As well as understanding that area is used in part to whole comparisons with regional models, students need to have a multiplicative sense of area rather than an additive appreciation. An additive appreciation of area typically results in counting units of area rather than multiplicatively subdividing a unit. Adding units of area will result in the whole growing, as in Figure 6.9.

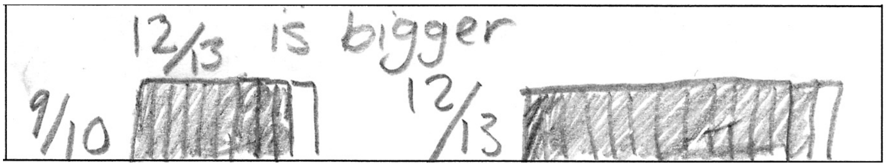

Figure 6.9. An example of an additive approach to representing larger denominators.

Another way of representing fractions is a part of a group, sometimes called the set model. This discrete model of fractions may also contribute to the development of a "growing whole" when comparing the size of fractions. For example, in comparing one-third and one-sixth, a student can draw and shade one of three compared to one of six, as in Figure 6.10.

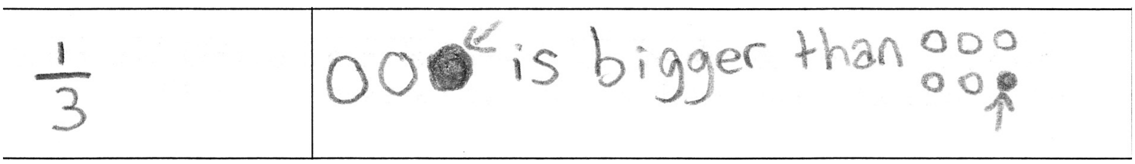

Figure 6.10. Which is bigger, one-third or one-sixth?

As seen earlier in Figure 6.3, students can also interpret what appear to be area models as discrete models. Although we rarely link these two markedly different representations of fractions (discrete and continuous) together, some students may do this in an unusual way. In Figure 6.11, the student represents the fractions as part of a collection (one-third as one of three) before using the larger collection as a common unit to re-present the fraction with the smaller denominator. While this response was quite unique, comparing fractions this way is very difficult. However, the recognition of the need for a common unit to enable comparison of the fractions is encouraging.

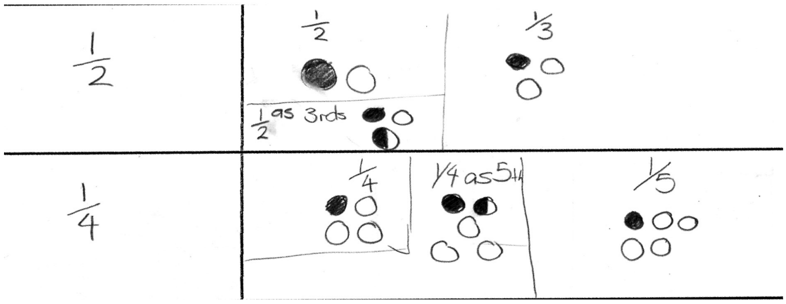

Figure 6.11. Combining discrete and continuous representations.

The elephant in the room: Fraction notation

In research, a fraction has been described as being interpreted with five views: part-whole, measure, ratio, operator and quotient (Behr, Harel, Post, & Lesh, 1992; Carraher, 1996; Kieren, 1976; Lamon, 2001). Although part-whole comparison is the most widely adopted view for the learner to start with, the part-whole view has been described as "the least valuable road into the system of rational numbers" (Lamon, 2001, p.163). So is there another, simpler way to think about fractions?

Fractions can be thought of as parts of objects or collections: one-half of an apple, three-quarters of a sandwich or one-third of a bag of marbles. Alternatively, fractions can be thought of as numbers: $\frac{1}{2}$, $\frac{3}{4}$ or $\frac{1}{3}$. Although we can say $\frac{1}{2}$ is greater than $\frac{1}{3}$ it does not make sense to suggest that one-half of an apple is greater than one-third of a bag of marbles. Yet when fraction notation is introduced in class it is usually as a way of recording a double count to describe fraction parts. First we count the number of parts shaded (usually of a regional model), next we count the total number of parts and then record the first count over the second count as a description of a fraction. Introducing fraction notation needs to pay explicit attention to the whole. One way to do this is to introduce students to fractions from the viewpoint of measuring quantities, and to focus initially on the dimension of length. The teaching activities used should minimise the possibility that students will focus on misleading or irrelevant aspects of the representations. For instance, to draw attention to what is being compared, vary this attribute to produce counter-examples. Instead of asking students to shade one-quarter of a pre-partitioned shape, they can be given a strip of paper, partitioned as shown in Figure 6.12 and asked to determine the shaded part as a fraction of the whole.

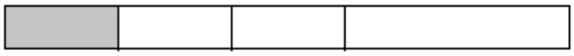

Figure 6.12. Distinguishing between one-fourth and one-fifth.

The link to fractional units of length is clearer for students than shading area (as a part-whole comparison) particularly if students do not know how to determine area. As teachers, it is critical that we are aware of the different interpretations students may make of fractions so as to glean those that students actually make. Focusing on units of length and providing carefully chosen counter-examples helps to limit the adoption of unintended features of fraction representations. Finally, fraction notation needs to be introduced very carefully as three-quarters of a sandwich is not the same as the number $\frac{3}{4}$. To get students thinking about the difference between the two, have them consider if one-quarter could ever be bigger than one-half.

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